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Solitons in (1,1)-supersymmetric massive sigma models

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ABSTRACT

We find the solitons of massive (1,1)-supersymmetric sigma models with target space the groups $SO(2)$ and $SU(2)$ for a class of scalar potentials and compute their charge, mass and moduli space metric. We also investigate the massive sigma models with target space any semisimple Lie group and show that some of their solitons can be obtained from embedding the $SO(2)$ and $SU(2)$ solitons.

1. Introduction

It has been known for some time that some of the non-perturbative aspects of field theories can be understood by examining the properties of their soliton solutions. This has recently been emphasized both in four-dimensional Yang-Mills theory and in string theory. Typically, field theories with solitons are those for which a lower bound can be established for the energy in terms of the values of their Noether and topological charges. The solitons are the classical solutions that saturate this bound. Therefore the solitons are also solutions of the Bogomol'nyi equations which enforce the saturation of the bound [1]. The Bogomol'nyi equations are usually first order differential equations in spacetime derivatives and simpler to solve than the field equations of a theory which are quadratic. Although a bound for the energy can be established for a large class of theories, it is better understood in the context of the supersymmetric ones. In this case, the bound is a consequence of the supersymmetry algebra, and the topological and the Noether charges that appear in the bound are central charges of this algebra [2].

Examples of such theories are the two-dimensional (p,q) -supersymmetric massive sigma models, with or without torsion [3,4,5]. Apart from the usual metric and torsion couplings of sigma models, the bosonic part of the action of these theories has a scalar potential. The energy is bounded from below by the value of various topological and Noether charges all of which are central charges of the (p,q) -supersymmetry algebra. Most of these models have solitons[★] that saturate the bound and interpolate between the different classical vacua; some explicit soliton solutions have been found in [6,7]. The class of massive sigma models with torsion includes the G -models, i.e. the $(1,1)$ -supersymmetric massive sigma models for which the target space is a group manifold G [7]. The bosonic part of the action of these models consists of the action of a Wess-Zumino-Witten model with target

★ We use the term 'soliton' for every solution of the Bogomol'nyi equations that has finite energy.

space a Lie group G and the scalar potential

$$V = \frac{m^2}{4} g^{ij} (2\kappa_A L_i^A + \partial_i a) (2\kappa_B L_j^B + \partial_j a) , \quad (1.1)$$

where g is the bi-invariant metric on G , L is the left invariant frame, m is a constant with dimension that of a mass, κ is a constant vector in the Lie algebra, $\mathcal{L}(G)$, of G , $A = 1, \dots, \dim \mathcal{L}(G)$, and

$$a = \frac{1}{n} \text{tr } k^n , \quad (1.2)$$

$n \in \mathbb{Z}$, $n \neq 0$, and $k \in G$. Note that the scalar potential (1.1) is specified by two parameters, the coupling constant κ and the integer n . Both the vacuum structure of these models and the properties of their solitons depend on the values of these two parameters.

In this paper, we shall systematically investigate the solutions of the Bogomol'nyi equations of G -models for which the target space is any semisimple compact Lie group G . It is sufficient to investigate those models for which $n > 0$ because the rest are related to the $n \geq 0$ ones by a field redefinition. It can also be arranged, without loss of generality, for the coupling constant κ to lie in the Cartan subalgebra of the group G . With this choice for κ , we shall show that all vacua lie on a maximal torus of the group G . We shall first solve the Bogomol'nyi equations of $SO(2)$ -models and show that the $SO(2)$ solitons generalise the sine-Gordon solitons; the latter are recovered in the limit $\kappa = 0$. We shall then give the soliton solutions of the $SU(2)$ model for $n = 1$ and $n = 2$ without posing any restriction on the coupling constant κ . In the $n = 1$ case, we shall find two classes of solitons. The first one is a set of static solitons that lie on a maximal torus of $SU(2)$ and are embeddings of the $SO(2)$ solitons. The second one is a set of time-dependent solutions that, apart from their asymptotic values, lie in the complement of the maximal torus of $SU(2)$ and include the solution given in [7]. In the $n = 2$ case, we shall show that all solitons which lie on the maximal torus are obtained from embedding of $SO(2)$ solitons into the $SU(2)$ model. In addition, we shall examine

the qualitative properties of the Bogomol'nyi equations for $SU(2)$ $n > 2$ models and support our conclusions with a numerical calculation for the $n = 3, 4, 5$ cases (Fig [1,2,3]). Finally, we shall show that the G -model for G any semisimple Lie group has static and time-dependent solutions. The former lie on the maximal torus of G while the latter lie on the complement of the maximal torus. Some of the G -model solitons are obtained from embedding $SO(2)$ and $SU(2)$ solitons.

The organisation of the paper is as follows: In section two, we shall introduce the Bogomol'nyi equations of G -models and discuss their general properties. In section three, we shall solve the Bogomol'nyi equations for $G = SO(2)$ and compute the charge, mass and moduli space metric of the solitons. In section four, we shall examine the general properties of the $SU(2)$ model and give all static solitons that lie on the maximal torus of $SU(2)$. In section five, we shall investigate the time-dependent solitons of the $SU(2)$ $n = 1$ model and give their charge, mass and moduli space metric. In section six, we shall examine the solitons of the $SU(2)$ $n = 2$ model and find that all of them are static and lie on the maximal torus. In section seven, we shall study the qualitative properties of the Bogomol'nyi equations for the $SU(2)$ $n > 2$ models. The solitons for models with any semisimple group G as target space will be investigated in section eight, and in section nine we shall give our conclusions and comment on the quantum mechanical properties of the solitons of G -models.

2. The Bogomol'nyi equations

Let \mathcal{N} be a Riemannian manifold with metric g , a locally defined two form b and a Killing vector X . The bosonic part of the action of (1,1)-supersymmetric massive sigma models with torsion H and scalar potential V [7] is

$$I = \int d^2x [(g + b)_{ij} \partial_+ \phi^i \partial_- \phi^j - V(\phi)] , \quad (2.1)$$

where ϕ is a map from the two-dimensional Minkowski spacetime, with light-cone co-ordinates (x^+, x^-) , into \mathcal{N} and the two-form b is the 'gauge potential' of the

torsion three-form H , $H_{ijk} = \frac{3}{2}\partial_{[i}b_{jk]}$. Furthermore, the scalar potential is

$$V = \frac{m^2}{4}g_{ij}(X^iX^j + u^iu^j) , \quad (2.2)$$

where the one-form u is orthogonal to the Killing vector X ($X^iu_i = 0$) and

$$X^iH_{ijk} = \partial_{[j}u_{k]} . \quad (2.3)$$

It is clear that the supersymmetric vacua of the theory are the points of the target space \mathcal{N} where both X and u vanish. The action of the (1,1)-supersymmetric massive sigma model is invariant under the transformations

$$\delta\phi^i = \eta X^i(\phi) \quad (2.4)$$

with associated charge

$$Q = \int dx (X_i \partial_t \phi^i + u_i \partial_x \phi^i) , \quad (2.5)$$

where η is an infinitesimal parameter and (t, x) are the co-ordinates of two-dimensional Minkowski spacetime. The charge Q is the charge that appears as central charge in the (1,1)-supersymmetry algebra [7]. The energy of the model is

$$E = \frac{1}{2} \int dx (g_{ij} \partial_t \phi^i \partial_t \phi^j + g_{ij} \partial_x \phi^i \partial_x \phi^j + V(\phi)) , \quad (2.6)$$

and the bound of E in terms of Q is

$$E \geq \frac{m}{2}|Q| . \quad (2.7)$$

We can split the charge Q into a Noether charge Q_N and a topological charge Q_T , $Q = Q_N + Q_T$. Although there is no natural way to split Q , we can use the Hodge

decomposition of u as a sum of an exact form α , a co-exact form β and a harmonic form γ , ($u = \alpha + \beta + \gamma$), with respect to the metric g to define

$$Q_T = \int (\alpha + \gamma)_i \partial_x \phi^i . \quad (2.8)$$

and $Q_N \equiv Q - Q_T$; Q_N and Q_T are separately conserved. Both the fundamental states and the solitons of the theory are charged with respect to the Noether charge Q_N but only the latter are charged with respect to the topological charge Q_T as well.

In the special case of G -models with scalar potential (1.1), we choose the couplings as follows: g is the bi-invariant metric

$$g_{ij} = \delta_{AB} L^A_i L^B_j = \delta_{AB} R^A_i R^B_j \quad (2.9)$$

on G , H is the bi-invariant closed three form

$$H_{ijk} = -\frac{1}{2} f_{ABC} L^A_i L^B_j L^C_k = -\frac{1}{2} f_{ABC} R^A_i R^B_j R^C_k \quad (2.10)$$

on G , X is the vector field

$$X^i = \kappa^A (L - R)^i_A , \quad (2.11)$$

and the one-form u is

$$u_i = \kappa_A (L + R)^A_i + \partial_i a , \quad (2.12)$$

where L is the left invariant frame of G , R is the right invariant frame of G , f are the structure constants of $\mathcal{L}(G)$, $A, B, C = 1, \dots, \dim \mathcal{L}(G)$ and $i, j, k = 1, \dots, \dim G$, κ is a vector in the Lie algebra of G as in (1.1) and a is given in (1.2).

The supersymmetric classical vacua of the action (2.1) are the solutions of the algebraic equation

$$2\kappa_A + \text{tr}(k^n t_A) = 0 \quad (2.13)$$

for $k \in G$, where $\{t_A; A = 1, \dots, \dim \mathcal{L}(G)\}$ is a basis of the Lie algebra of the group G . The charge Q (2.5) of G -models is

$$Q = \int dx (\kappa_A (L - R)_i^A \partial_t \phi^i + \kappa_A (L + R)_i^A \partial_x \phi^i + \partial_x a) . \quad (2.14)$$

For G abelian, $L = R$ and $\kappa_A (L + R)^A$ is a closed one-form, so $Q_N = 0$ and

$$Q_T = Q = \int dx (2\kappa_A L_i^A \partial_x \phi^i + \partial_x a) . \quad (2.15)$$

However, for G semisimple, the one-form $u - da$ is co-exact and the topological charge Q_T (2.8) is simply

$$Q_T = \int dx \partial_x a . \quad (2.16)$$

To find the mass and charge of the bosonic fundamental states of G -models, we use the background field method (see for example [8]) to linearise the theory around a classical supersymmetric vacuum k_0 . The mass-square and charge matrices written in the left-invariant frame basis are then

$$\mathcal{M}_{AB} = \frac{m^2}{4} (\kappa^C \kappa^D f^E{}_{AC} f_{EBD} + n^2 \sum_C \text{tr}(t_{(A} t_C) k_0^n) \text{tr}(t_{B)} t_C k_0^n)) , \quad (2.17)$$

and

$$\mathcal{Q}_{AB} = -\kappa^C f_{ABC} , \quad (2.18)$$

respectively. The eigenvalues of \mathcal{M} are the square of the masses of the fundamental states and the eigenvalues of \mathcal{Q} are their charges. To do this computation, we have used (2.3) to show that $du = 0$ evaluated at any supersymmetric vacuum. The

fundamental states parallel to the zero modes of the charge-matrix (2.18) have zero Noether charge. The zero modes of \mathcal{Q} are orthogonal to the tangent vectors of the coadjoint orbit of G determined by κ and therefore the directions that carry the charge are those along the coadjoint orbit; in fact \mathcal{Q} is the symplectic form of the coadjoint orbit. The mass of the fundamental states need not saturate the bound (2.7) since

$$\mathcal{M} \geq \frac{m^2}{4} \mathcal{Q} \mathcal{Q}^t \quad (2.19)$$

due to the presence of the second term in the mass-square-matrix (2.17). So the fundamental states obey the bound as expected for supersymmetric theories but they are not necessarily BPS states. Apart from the bosonic fundamental states, the (1,1)-supersymmetric sigma models have also fermionic ones which can be investigated in a similar way; for the purpose of this paper we shall restrict our attention to the bosonic fundamental states.

The classical configurations that saturate the bound (2.7) satisfy the following Bogomol'nyi type equations:

$$\begin{aligned} \partial_t k(x, t) &= \mp \frac{m}{2} (\kappa k - k \kappa) \\ \partial_x k(x, t) &= \mp \frac{m}{2} (\kappa k + k \kappa + k \sum_A t_A \text{tr}(t_A k^n)) . \end{aligned} \quad (2.20)$$

The first of these equations can be easily solved by setting

$$k(x, t) = \exp \left(\mp \frac{m}{2} \kappa t \right) k(x) \exp \left(\pm \frac{m}{2} \kappa t \right) . \quad (2.21)$$

Substituting this into the second equation of (2.20), it reduces to

$$\frac{d}{dx} k(x) = \mp \frac{m}{2} (\kappa k(x) + k(x) \kappa + k(x) \sum_A t_A \text{tr}(t_A k^n(x))) . \quad (2.22)$$

This is a non-linear but ordinary differential equation and the investigation of its solutions for various choices of G and n will be our task in the rest of the paper.

It is sufficient to investigate (2.22) for $n > 0$ because the transformation

$$k \rightarrow k^{-1}, \quad x \rightarrow -x \quad (2.23)$$

acting on a model with coupling constants g, b, κ, m and $a = \frac{1}{n} \text{tr } k^n$ transforms it to a model with coupling constants g, b, κ, m and $a = -\frac{1}{n} \text{tr } k^{-n}$. Therefore the solitons of models with $n < 0$ can be obtained from the solitons of models with $n > 0$ by acting on the latter with (2.23). In addition, the Lagrangian (2.1), the field equations and the Bogomol'nyi equations of the (1,1)-supersymmetric sigma model are invariant under the sigma model symmetry

$$k(x, t) \rightarrow h k(x, t) h^{-1} \quad \kappa \rightarrow h \kappa h^{-1}, \quad (2.24)$$

where $h \in G$. Although sigma model symmetries are not associated with conserved charges, we can use (2.24) to choose, without loss of generality, the coupling constant κ in the Cartan subalgebra of $\mathcal{L}(G)$. This choice of κ simplifies both the Bogomol'nyi equations (2.20) and the equation for the vacua (2.13) of the model. In particular, we shall show in chapter eight that it can be arranged for the vacua to lie in the maximal torus of the group G .

The solutions of the Bogomol'nyi equations may depend on several moduli parameters. In that case, as for BPS monopoles [9], one can define a metric on the moduli space of solutions as follows:

$$ds^2 = \frac{1}{2} \int dx g_{ij} d\phi^i d\phi^j, \quad (2.25)$$

where the differential on ϕ denotes variation with respect to the moduli parameters. The soliton solutions of the Bogomol'nyi equations (2.20) preserve 1/2 of the supersymmetry and therefore their effective theory is an $N = 1$ supersymmetric one-dimensional sigma model with target space the moduli space and metric the moduli metric (2.25).

3. The $SO(2)$ model

The simplest of the G -models is the one with the group $SO(2)$ as target space. Since the target space is one dimensional, the antisymmetric tensor coupling b is identically zero. A parameterisation of the group manifold $SO(2)$ is

$$k = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad (3.1)$$

and a basis in $\mathcal{L}(SO(2))$ is

$$t_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (3.2)$$

where $0 \leq \theta < 2\pi$. To find the classical vacua of the $SO(2)$ model, we use the same symbol κ for the vector κ and its component along t_1 and substitute (3.1) into the equation for the vacua (2.13) to find

$$\sin(n\theta) = \kappa. \quad (3.3)$$

For $|\kappa| > 1$, the model does not have any supersymmetric vacua and therefore supersymmetry is spontaneously broken. For $0 \leq \kappa \leq 1$, the supersymmetric vacua of the $SO(2)$ model are

$$\theta[\tilde{\ell}, \pm] = \frac{1}{n} \left(\pm \theta_0 + \frac{1 \mp 1}{2} \pi + 2\pi \tilde{\ell} \right), \quad (3.4)$$

where $\tilde{\ell}$ is an integer, $0 \leq \tilde{\ell} < n$, and $\sin \theta_0 = \kappa$, $0 \leq \theta_0 < \frac{\pi}{2}$; the angle θ_0 vanishes for $\kappa = 0$. This $2n$ vacua can be ordered by the size of the angle θ as follows:

$$\theta[0, +] < \cdots < \theta[\tilde{\ell}, +] < \theta[\tilde{\ell}, -] < \theta[\tilde{\ell} + 1, +] < \cdots < \theta[n - 1, -]. \quad (3.5)$$

For $\kappa = 1$, $\theta_0 = \frac{\pi}{2}$, there are only n supersymmetric vacua (without counting

multiplicities) and the equation for the vacua (3.4) can be rewritten as

$$\theta[\tilde{\ell}] = \frac{\pi}{n} \left(\frac{1}{2} + 2\tilde{\ell} \right) . \quad (3.6)$$

We can order these vacua with respect to the value of $\tilde{\ell}$. We shall not investigate the models with $-1 \leq \kappa < 0$ since it is easy to generalise the results obtained in the models with $0 \leq \kappa \leq 1$ to this case.

All $SO(2)$ solitons are static. To find them, we must solve (2.22) which rewritten in the co-ordinates (3.1) becomes

$$\frac{d}{dx}\theta = \pm m \left(-\kappa + \sin(n\theta) \right) . \quad (3.7)$$

We shall consider the following three cases: (i) For $0 < \kappa < 1$, a linearisation of the equation (3.7) with the plus sign around the vacua $\theta[\tilde{\ell}, \pm]$ reveals that the vacua $\theta[\tilde{\ell}, +]$ are ‘sources’ while the vacua $\theta[\tilde{\ell}, -]$ are ‘sinks’. (If we choose the minus sign in (3.7) then the sources and sinks will be interchanged). This suggests that there should be solutions that interpolate between neighbouring sources and sinks. Indeed the solutions that interpolate between the vacua $\theta[\ell, +]$ and $\theta[\ell, -]$ are

$$\theta(x) = \frac{2}{n} \left(\arctan \left[\frac{1}{\kappa} \left[1 + \sqrt{1 - \kappa^2} \tanh \left(\pm \frac{nm}{2} \sqrt{1 - \kappa^2} (x - x_0) \right) \right] \right] + \ell\pi \right) , \quad (3.8)$$

where $0 \leq \ell < n$, $\ell \in \mathbb{N}$, and x_0 is an integration constant due to the translation invariance of the underlying model. In addition, the solutions

$$\theta(x) = \begin{cases} \frac{2}{n} \left(\arctan \left[\frac{1}{\kappa} \left[1 + \sqrt{1 - \kappa^2} \coth \left(\pm \frac{nm}{2} \sqrt{1 - \kappa^2} (x - x_0) \right) \right] \right] \right. \\ \quad \left. + (\ell + 1)\pi \right) , & \pm x < 0 \\ \frac{2}{n} \left(\arctan \left[\frac{1}{\kappa} \left[1 + \sqrt{1 - \kappa^2} \coth \left(\pm \frac{nm}{2} \sqrt{1 - \kappa^2} (x - x_0) \right) \right] \right] \right. \\ \quad \left. + \ell\pi \right) , & \pm x > 0 \end{cases} \quad (3.9)$$

interpolate between the vacua $\theta[\ell, -]$ and $\theta[\ell + 1, +]$, for $0 \leq \ell < n - 1$, and the vacua $\theta[n - 1, -]$ and $\theta[0, +]$ for $\ell = n - 1$. We remark that the solutions (3.9)

are continuous and differentiable at $x = 0$. The solitons are those solutions in (3.8) and in (3.9) with the plus sign whereas the anti-solitons are those with the minus sign. We note that the $SO(2)$ solitons of (3.8) and (3.9) interpolate between all neighbouring pairs of vacua of the model. (ii) For $\kappa = 0$, the equation (3.7) becomes that of the sine-Gordon theory. The solutions are

$$\theta = \zeta \frac{2}{n} \left(\arctan \left[\exp \left(\pm mn(x - x_0) \right) \right] + \ell\pi \right) \quad (3.10)$$

for $0 \leq \ell < n$, $\ell \in \mathbb{Z}$ and $\zeta = \pm$. (iii) For $\kappa = 1$ and $n > 1$, the solutions that interpolate between the vacua $\theta[\ell]$ and $\theta[\ell + 1]$, for $0 \leq \ell < n - 1$, and the vacua $\theta[n - 1]$ and $\theta[0]$ for $\ell = n - 1$, are

$$\theta = \frac{2}{n} \left(\arctan \left[\mp mn(x - x_0) \right] - \frac{\pi}{4} + \ell\pi \right). \quad (3.11)$$

These are $2n$ solutions, n solitons and n anti-solitons, interpolating between all neighbouring pairs of vacua as in the previous two cases.

For $SO(2)$ models, the charge Q (2.14) is equal to the topological charge Q_T (2.8). The value of Q for the solitons is

$$Q = \mp \frac{4}{n} \left[\sqrt{1 - \kappa^2} + \kappa \arcsin(\kappa) + \xi \frac{\kappa}{2} \pi \right], \quad (3.12)$$

where ξ is either minus or plus, ($\xi = \mp$); Q for $\xi = -$ is the charge of (3.8) solutions and Q for $\xi = +$ is the charge of (3.9) solutions. From this, it is straightforward to compute the mass of the solitons to find[★]

$$E = \frac{2m}{n} \left[\sqrt{1 - \kappa^2} + \kappa \arcsin(\kappa) + \xi \frac{\kappa}{2} \pi \right]. \quad (3.13)$$

The $SO(2)$ solitons (3.8) and (3.9) have one moduli parameter x_0 . Using (2.25),

★ We have identified the energy with the mass of a soliton configuration.

the metric on the moduli space is

$$ds^2 = \frac{m}{n} \left[\sqrt{1 - \kappa^2} + \kappa \arcsin(\kappa) + \xi \frac{\kappa}{2} \pi \right] dx_0^2 = \frac{E}{2} dx_0^2 . \quad (3.14)$$

The charge, the mass and the moduli metric of the solitons (3.10) are derived by setting $\kappa = 0$ in the above corresponding expressions.

Finally, the charge, the mass and the moduli metric of the solutions (3.11) of models with $\kappa = 1$ are

$$Q = \mp \frac{4}{n} \pi , \quad (3.15)$$

$$E = \frac{2m}{n} \pi , \quad (3.16)$$

and

$$ds^2 = \frac{m}{n} \pi dx_0^2 = \frac{E}{2} dx_0^2 , \quad (3.17)$$

respectively.

4. The $SU(2)$ model

A parameterisation of $SU(2)$ in terms of 2×2 matrices is

$$k = \begin{pmatrix} M + iN & -U + iW \\ U + iW & M - iN \end{pmatrix} ; \quad M^2 + N^2 + U^2 + W^2 = 1 , \quad (4.1)$$

and a choice of basis in the Lie algebra $\mathcal{L}(SU(2))$ of $SU(2)$ is

$$t_1 = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad t_2 = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad t_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \quad (4.2)$$

Using the observation in section 2 that the Bogomol'nyi equations are invariant under the adjoint action of G on both the fields and the coupling constant κ , we

can choose, without loss of generality, κ in the Cartan subalgebra of $\mathcal{L}(SU(2))$, i.e.

$$\kappa = \kappa t_1 . \quad (4.3)$$

We have denoted both the vector κ and its component along t_1 with the same symbol. The distinction between the two will be clear from the context.

The vacua of the theory (2.13) in the above parameterisation are the solutions of the equation

$$k^n = \begin{pmatrix} \mp\sqrt{1-\kappa^2} + i\kappa & 0 \\ 0 & \mp\sqrt{1-\kappa^2} - i\kappa \end{pmatrix} , \quad (4.4)$$

in terms of k , $n \in \mathbb{N}$. It is clear that for $|\kappa| > 1$, there are no supersymmetric vacua and so supersymmetry is spontaneously broken. For $|\kappa| < 1$ there are $2n$ supersymmetric vacua, and for $|\kappa| = 1$ there are n supersymmetric vacua (without counting multiplicities in the latter case). We shall restrict our attention to $\kappa \geq 0$ because it is straightforward to generalise the results below to $SU(2)$ models with $\kappa < 0$. We remark that linearising the theory around the vacua (4.4), the mass-square-matrix and charge-matrix of the fundamental states of the $SU(2)$ model are

$$\mathcal{M} = m^2 \begin{pmatrix} 1 - \kappa^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad (4.5)$$

and

$$\mathcal{Q} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2\kappa \\ 0 & 2\kappa & 0 \end{pmatrix} , \quad (4.6)$$

respectively. Clearly for $|\kappa| < 1$, $\mathcal{M} > \frac{m^2}{4} \mathcal{Q} \mathcal{Q}^t$ and the fundamental states are not BPS states. However, for $|\kappa| = 1$, $\mathcal{M} = \frac{m^2}{4} \mathcal{Q} \mathcal{Q}^t$ and all the fundamental states become BPS with one of them massless.

The equations (2.22) in this parameterisation for the $SU(2)$ model are

$$\begin{aligned}
\frac{d}{dx}M \pm m[-\kappa N + (1 - M^2)\tilde{M}_n] &= 0 \\
\frac{d}{dx}N \pm m[\kappa M - NM\tilde{M}_n] &= 0 \\
\frac{d}{dx}U \mp mUM\tilde{M}_n &= 0 \\
\frac{d}{dx}W \mp mWM\tilde{M}_n &= 0 ,
\end{aligned} \tag{4.7}$$

where

$$\tilde{M}_n = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2i+1} (-1)^i M^{n-2i-1} (1 - M^2)^i . \tag{4.8}$$

There are three independent equations in (4.7) since the fourth one is implied from the restriction that M, N, U, W lie on a 3-sphere. Moreover from the last two equations of (4.7), it is easy to see that

$$U = \lambda W , \tag{4.9}$$

where λ is a real constant. So we only need to determine M, N from the first two equations in (4.7); these equations will be studied in the next three chapters for various values of n .

It is convenient for reasons that will become apparent later to parameterise $SU(2)$ in angular coordinates as follows:

$$\begin{aligned}
M &= \cos \theta \\
N &= \sin \theta \cos \phi \\
U &= \sin \theta \sin \phi \cos \psi \\
W &= \sin \theta \sin \phi \sin \psi ,
\end{aligned} \tag{4.10}$$

where $0 \leq \theta, \phi < \pi$ and $0 \leq \psi < 2\pi$. In this parameterisation, the equation (4.4)

for the vacua can be written as

$$\begin{aligned}\sin(n\theta) &= \pm\kappa \\ \cos\phi &= \pm 1 .\end{aligned}\tag{4.11}$$

Note that the vacua of the theory lie in $\phi = 0$ and $\phi = \pi$ semi-circle subspaces of $SU(2)$, which are joined at $(\theta, \phi) = (0, 0)$ and $(\theta, \phi) = (0, \pi)$, and at $(\theta, \phi) = (\pi, 0)$ and $(\theta, \phi) = (\pi, \pi)$ to a circle in $SU(2)$; in fact this circle is a maximal torus of $SU(2)$. Solving (4.11) for $0 \leq \kappa < 1$, we find that the $2n$ vacua of the model are

$$(\theta, \phi) = \begin{cases} \left(\frac{1}{n}(\pm\theta_0 + \frac{1\mp 1}{2}\pi + 2\pi\tilde{\ell}), 0 \right), & 0 \leq \tilde{\ell} \leq \left[\frac{n-1}{2}\right] \\ \left(\frac{1}{n}(\pm\theta_0 + \frac{1\mp 1}{2}\pi + 2\pi(\tilde{\ell} + \frac{1}{2})), \pi \right), & 0 \leq \tilde{\ell} < \left[\frac{n}{2}\right], \end{cases}\tag{4.12}$$

where $0 \leq \theta_0 < \frac{\pi}{2}$ and $\sin\theta_0 = \kappa$. For $\kappa = 0$, θ_0 vanishes and the expression for the vacua becomes

$$\begin{aligned}(\theta, \phi) &= \left(\frac{\ell}{n}\pi, 0 \right) \\ (\theta, \phi) &= \left(\frac{\ell+1}{n}\pi, \pi \right),\end{aligned}\tag{4.13}$$

where $0 \leq \ell < n$, $\ell \in \mathbb{N}$. For $\kappa = 1$, the n vacua of the theory (without counting multiplicities) can be found by setting $\theta_0 = \pi/2$ in (4.12).

The equations (2.22) written in the angular parameterisation are

$$\begin{aligned}\frac{d}{dx}\theta &= \pm m[-\kappa \cos\phi + \sin(n\theta)] \\ \frac{d}{dx}\phi &= \pm m\kappa \cot\theta \sin\phi \\ \frac{d}{dx}\psi &= 0 .\end{aligned}\tag{4.14}$$

The third equation above implies that the angle ψ is constant; ψ is related to the ratio of U, W as $\cot\psi = \lambda$. So it remains to solve the first two equations of (4.14) for θ and ϕ for the various values of n and κ .

To find a class of solutions for the equations (4.14) above, we set ϕ to its vacuum values, i.e. $\phi = 0, \pi$. This choice for ϕ solves the second of the equations (4.14), so it remains to solve the first for θ . Since we have set $\phi = 0, \pi$, the solutions lie on the maximal torus of $SU(2)$, and they are best described by extending the range of θ from $0 \leq \theta < \pi$ in (4.14) to $0 \leq \theta < 2\pi$. Then, the first equation in (4.14) is precisely the equation (3.7) that we have encountered in the $SO(2)$ model. Moreover, the vacua (4.12) of the $SU(2)$ model can be identified with the vacua (3.4) of the $SO(2)$ model as follows:

$$\theta[\ell, \pm] = \begin{cases} \left(\frac{1}{n}(\pm\theta_0 + \frac{1\mp 1}{2}\pi + 2\pi\ell), 0 \right), & 0 \leq \ell \leq \left[\frac{n-1}{2} \right] \\ \left(\frac{1}{n}(\pm\theta_0 + \frac{1\mp 1}{2}\pi + 2\pi(\tilde{\ell} + \frac{1}{2})), \pi \right), & \left[\frac{n-1}{2} \right] < \ell < n, \end{cases} \quad (4.15)$$

where $\tilde{\ell} = n - \ell - 1$. Furthermore, as can be easily seen from (2.21), the solitons of the $SU(2)$ model that lie on the maximal torus are static. From this we conclude that the solitons of the $SU(2)$ model that lie on the maximal torus are embeddings of the solitons of the $SO(2)$ model found in the previous section with

$$\theta \rightarrow \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \quad (4.16)$$

for the associated value of n . The class of solutions that we have described includes those for which κ vanishes. This is because, if $\kappa = 0$, the second equation in (4.14) implies that ϕ is constant and the asymptotic behaviour of the solitons requires that it should be set to its vacuum values.

Note from (2.15) and (2.16) that the definition of the topological charge of the G -models with G semisimple is different from the definition of the topological charge of the $SO(2)$ models. The value of the topological charge Q_T of any soliton of the $SU(2)$ model, or any G -model with G semisimple, interpolating between the two vacua (θ, ϕ) and (θ', ϕ') is $Q_T = a(\theta', \phi') - a(\theta, \phi)$ while for the $SO(2)$ solitons $Q_T = Q$. Therefore the value of the topological charge of the $SO(2)$ solitons and the

value of the topological charge of their embeddings as $SU(2)$ solitons are different. Apart from the $SU(2)$ solitons that lie on the maximal torus and we have examined above, the $SU(2)$ model has other solitons that interpolate between different vacua but otherwise lie entirely outside the maximal torus of $SU(2)$. We shall investigate these solitons in the following three sections.

5. The $SU(2)$ $n=1$ model

As we have seen in the previous section, there are static solitons that interpolate between the two vacua of $SU(2)$ $n = 1$ model with $0 \leq \kappa < 1$ and lie entirely on the maximal torus of $SU(2)$. Here, we shall find another class of solitons for this model that interpolate between the two vacua but otherwise lie in the *complement* of the maximal torus of $SU(2)$. The equations (4.7) for M, N in the $n = 1$ case are

$$\begin{aligned} \frac{d}{dx}M \pm m[-\kappa N + (1 - M^2)] &= 0 \\ \frac{d}{dx}N \pm m[\kappa M - NM] &= 0 . \end{aligned} \tag{5.1}$$

Following [7], we can find a class of solitons by setting $N = \kappa$. These solutions[★] are

$$\begin{aligned} M &= \sqrt{1 - \kappa^2} \tanh \left(\mp m \sqrt{1 - \kappa^2} (x - x_0) \right) \\ N &= \kappa \\ U &= \sqrt{1 - \kappa^2} \cosh^{-1} \left(m \sqrt{1 - \kappa^2} (x - x_0) \right) \cos \psi \\ W &= \sqrt{1 - \kappa^2} \cosh^{-1} \left(m \sqrt{1 - \kappa^2} (x - x_0) \right) \sin \psi , \end{aligned} \tag{5.2}$$

where x_0, ψ are the modular parameters of the solution, $\cot \psi = \lambda$. It is clear that these solitons lie in the complement of the maximal torus since U and W do not vanish for any value of x unless x goes to infinity. The topology of the moduli space is $S^1 \times \mathbb{R}$: The modular parameter x_0 is due to the translational invariance of the underlying theory while the modular angular parameter ψ is due to the

★ These solutions are related to those of [7] by a conjugation.

charge Q given in (2.14). These are reminiscent to the modular parameters of BPS monopoles in four-dimensions (see for example [10]).

However, there is a more general class of $SU(2)$ solutions for which $N \neq \kappa$. To find these new solitons, we use the angular parameterisation and the equations (4.14) for $n = 1$ become

$$\begin{aligned}\frac{d}{dx}\theta &= \pm m[-\kappa \cos \phi + \sin(\theta)] \\ \frac{d}{dx}\phi &= \pm m\kappa \cot \theta \sin \phi .\end{aligned}\tag{5.3}$$

As it can be seen from (4.12), the $SU(2)$ $n = 1$ model has two vacua $(\theta_0, 0)$ and $(\pi - \theta_0, 0)$. We remark that a linearisation of (5.3) at these two vacua reveals that one of them is a source while the other is a sink. The differential equations (5.3) is a Hamiltonian flow for the function

$$\alpha(\theta, \phi) = -\frac{1}{\sin \phi \sin \theta} + \frac{1}{\kappa} \cot \phi \tag{5.4}$$

with symplectic form

$$\Omega = \pm \frac{1}{(m\kappa) \sin^2 \phi \sin \theta} d\theta \wedge d\phi . \tag{5.5}$$

Since α is preserved by the flow, we can rewrite (5.4) as

$$\sin \theta = \frac{\kappa}{\cos \phi - \alpha \kappa \sin \phi} , \tag{5.6}$$

where α is a real constant. The solutions of (5.3) in the x parameterisation are

$$\begin{aligned}\cot \phi \equiv z &= \frac{\kappa}{2\sqrt{1-\kappa^2}} \exp \left[\pm m\sqrt{1-\kappa^2}(x-x_0) \right] \\ &\quad \left(\left[\exp \left[\mp m\sqrt{1-\kappa^2}(x-x_0) \right] + \frac{\alpha}{\sqrt{1-\kappa^2}} \right]^2 + 1 - \alpha^2 \right) \\ \cos \theta &= \begin{cases} \frac{\sqrt{(z-\alpha\kappa)^2 - \kappa^2(1+z^2)}}{z-\alpha\kappa} , & -\infty < x < x_{min} \\ -\frac{\sqrt{(z-\alpha\kappa)^2 - \kappa^2(1+z^2)}}{z-\alpha\kappa} , & x_{min} < x < +\infty , \end{cases}\end{aligned}\tag{5.7}$$

where at $x = x_{min}$, $z(x)$ takes its absolute minimum value. To verify that (5.7) solves the equations (5.3), we differentiate z with respect to x and use the equation

for ϕ to eliminate the derivative of ϕ from the expression. The x dependence can also be eliminated by inverting z to express the exponential of x in terms of z . However this equation is quadratic and there are two possible solutions distinguished by a sign. To satisfy the equations (5.3), one has to choose for $-\infty < x < x_{min}$ the solution with the plus sign and for $x_{min} < x < +\infty$ the solution with the minus sign together with the corresponding expression for $\cos \theta$ in (5.7).

To find the solitons of the $SU(2)$ $n = 1$ model, we substitute (5.7) in (2.21) and observe that they are *time-dependent*. For $\alpha = 0$, (5.7) reduces to the solution (5.2) given above. Moreover, after the redefinition $x_0 \rightarrow x_0 \pm \frac{1}{m\sqrt{1-\kappa^2}} \log(|\alpha|^{\frac{1-\sqrt{1-\kappa^2}}{\sqrt{1-\kappa^2}}})$, (5.7) reduces in the limit $\alpha \rightarrow \pm\infty$ to the embedded $SO(2)$ solitons found in the previous section. The charge and the mass of the solutions (5.7) are

$$Q = \mp 4\sqrt{1-\kappa^2}e^\sigma, \quad (5.8)$$

and

$$E = 2m\sqrt{1-\kappa^2}e^\sigma, \quad (5.9)$$

respectively, where

$$e^\sigma = 1 + \frac{\alpha\kappa^2}{\sqrt{1-\kappa^2}\sqrt{1+\alpha^2\kappa^2}} \left[-\frac{\pi}{2} + \arctan\left(\frac{\alpha\kappa^2}{\sqrt{1-\kappa^2}\sqrt{1+\alpha^2\kappa^2}}\right) \right]. \quad (5.10)$$

The constant α is not a modular parameter because the mass and the charge depend upon it. For example, the mass of the $\alpha = 0$ solutions (5.2) is[★]

$$E = 2m\sqrt{1-\kappa^2}. \quad (5.11)$$

Note that the mass (5.9) of the solitons as a function of α has critical points at $\alpha \rightarrow \pm\infty$. It turns out that $E(+\infty)$ is the absolute minimum and is the value

★ This corrects the expression for the mass of these solutions in [5].

of the mass of the embedded static solution (3.8) for $n = 1$, while $E(-\infty)$ is the absolute maximum and is the value of the mass of the embedded static solution (3.9) for $n = 1$. This is in agreement with the corresponding limits of the solution (5.7) mentioned above. The moduli space of the solutions (5.7) is again a cylinder with co-ordinates (x_0, ψ) and with metric

$$ds^2 = e^\sigma \frac{\sqrt{1 - \kappa^2}}{m(1 + \alpha^2 \kappa^2)} \left[m^2(1 - \kappa^2 + \alpha^2 \kappa^2) dx_0^2 + d\psi^2 \right]. \quad (5.12)$$

We remark that there is a transformation of the moduli co-ordinates $(x_0, \psi) \rightarrow (y, \chi)$ such that the moduli space metric above can be written in the form $ds^2 = \frac{E}{2}(dy^2 + d\chi^2)$.

6. The $SU(2)$ $n=2$ model

To show that all solitons of the $SU(2)$ $n = 2$ model lie on the maximal torus of $SU(2)$, we begin from the equations (4.7) for M, N with $n = 2$

$$\begin{aligned} \frac{d}{dx} M \pm m[-\kappa N + 2(1 - M^2)M] &= 0 \\ \frac{d}{dx} N \pm m[\kappa M - 2NM^2] &= 0. \end{aligned} \quad (6.1)$$

We then use the field redefinitions

$$X = \frac{N}{M}, \quad Y = \frac{N}{\sqrt{1 - M^2}} \quad (6.2)$$

to simplify (6.1) to

$$\begin{aligned} \frac{d}{dx} X \pm m[\kappa(1 + X^2) - 2X] &= 0 \\ \frac{d}{dx} Y \pm m\kappa X^{-1} Y[1 - Y^2] &= 0. \end{aligned} \quad (6.3)$$

The solutions in terms of X, Y for $0 < \kappa < 1$ are

$$\begin{aligned} X &= \frac{1}{\kappa} \left(1 + \sqrt{1 - \kappa^2} \tanh \left(\pm m \sqrt{1 - \kappa^2} (x - x_0) \right) \right) \\ \frac{Y^2}{1 - Y^2} &= \rho^{-1} X^2 \exp \left(\mp 2m(x - x_0) \right) \cosh^2 \left(m \sqrt{1 - \kappa^2} (x - x_0) \right), \end{aligned} \quad (6.4)$$

and

$$\begin{aligned} X &= \frac{1}{\kappa} \left(1 + \sqrt{1 - \kappa^2} \coth \left(\pm m \sqrt{1 - \kappa^2} (x - x_0) \right) \right) \\ \frac{Y^2}{1 - Y^2} &= \rho^{-1} X^2 \exp \left(\mp 2m(x - x_0) \right) \sinh^2 \left(m \sqrt{1 - \kappa^2} (x - x_0) \right), \end{aligned} \quad (6.5)$$

where ρ, x_0 are real constants ($\rho \geq 0$). It remains to examine whether or not these solutions interpolate between the different vacua of the theory. As we have already mentioned in section four, the $SU(2)$ $n = 2$ model has four vacua. It turns out that the solutions (6.5) interpolate between two different vacua only for $\rho = 0$, in which case $U = W = 0$. This implies that all solitons of the $SU(2)$ $n = 2$ model lie on the maximal torus. The same applies for all solitons of the $SU(2)$ $n = 2$ model with $\kappa = 1$.

To explain why there are no solitons of the $SU(2)$ $n = 2$ model that lie in the complement of the maximal torus of $SU(2)$, let us study the vacuum structure of this model in more detail. A linearisation of the equations (4.14) around the vacua (4.12) reveals that $(\frac{\theta_0}{2}, 0)$ and $(\frac{-\theta_0 + 2\pi}{2}, \pi)$ are sources, and $(\frac{\theta_0 + \pi}{2}, \pi)$ and $(\frac{-\theta_0 + \pi}{2}, 0)$ are saddles (for the Bogomol'nyi equations with the plus sign). Solving the equations (4.14) with $n = 2$ for θ and ϕ by setting $\phi = 0, \pi$, we find precisely the solitons of the $SU(2)$ $n = 2$ model that lie on the maximal torus of $SU(2)$ and therefore are embeddings of the $SO(2)$ solitons. These solitons interpolate between the source ($\phi = 0$) and the saddle ($\phi = 0$), the saddle ($\phi = 0$) and the source ($\phi = \pi$), the source ($\phi = \pi$) and the saddle ($\phi = \pi$), and the saddle ($\phi = \pi$) and the source ($\phi = 0$). Any additional solutions should interpolate either between the two saddles or between the saddles and the sources. But the only directions

from the two saddles which are not connected to another vacuum are those that point outwards. Therefore no soliton can exist that starts from one saddle to go to the other or from a saddle to go to a source. In fact, the outward directions from the saddles and some of the directions from the sources are connected to a sink at

$$(\theta, \phi) = \left(\frac{\pi}{2}, \frac{\pi}{2}\right), \quad (6.6)$$

which is a fixed point of the flow described by the equations (4.14), $n = 2$, but *not* a vacuum of the theory. We shall conjecture in the next section that a similar behaviour occurs in all $SU(2)$ models with n even.

7. The $SU(2)$ $n > 2$ models

The equations (4.7) or (4.14) for $SU(2)$ models with $n > 2$ and $\kappa \neq 0$ are rather involved and we have not been able to find the solitons that lie in the complement of the maximal torus of $SU(2)$. Another way to proceed is to investigate the qualitative properties of the solutions of these equations by linearising them about the vacua of the theory. This will allow us to find the pairs of vacua that are connected by solitons. The results of this analysis for $n = 3, 4, 5$ have been confirmed with a numerical computation (see Figs. [1,2,3]).

It is sufficient to examine the linearisation properties of the equations (4.14) in the interval $0 \leq \theta < \frac{\pi}{2}$. This is because the equations (4.14) are invariant under the discrete symmetries

$$(\theta, \phi, x) \rightarrow (\pi - \theta, \pi - \phi, x) \quad (7.1)$$

for n even, and

$$(\theta, \phi, x) \rightarrow (\pi - \theta, \phi, -x) \quad (7.2)$$

for n odd, which can be used to extend the analysis to the whole range of θ , ($\theta \in [0, \pi)$). Another property of the equations (4.14) for n even is that, apart from the vacua of the theory, there is another fixed point at $(\theta, \phi) = (\frac{\pi}{2}, \frac{\pi}{2})$.

Next we order the vacua (θ, ϕ) in (4.12) with respect to the value of θ , and denote these values by θ_i , $i = 1, \dots, 2n$, i.e. $\theta_1 < \dots < \theta_i < \theta_{i+1} < \dots < \theta_{2n}$. For θ in $[0, \pi/2)$, a linearisation of (4.14) with the plus sign reveals that the vacua θ_{4q+1} are sources, the vacua θ_{4q+3} are sinks and the rest are saddle points. Observe that the sources θ_{4q+1} and the saddles θ_{4q+2} lie at the $\phi = 0$ semicircle, while the sinks θ_{4q+3} and the saddles θ_{4q+4} lie at the $\phi = \pi$ semicircle. First, we shall consider the solutions interpolating amongst vacua θ_i in the interval $[0, \pi/2)$. It is expected that there should be a one parameter family of solitons that interpolate between the source θ_{4q+1} and the sink θ_{4q+3} , and the same source and the sink θ_{4q-1} . There also should be a unique soliton interpolating between the source θ_{4q+1} and the saddle θ_{4q} , and the sink θ_{4q+3} and the saddle θ_{4q+2} . In addition to these solutions, we have the solitons that lie on the maximal torus.

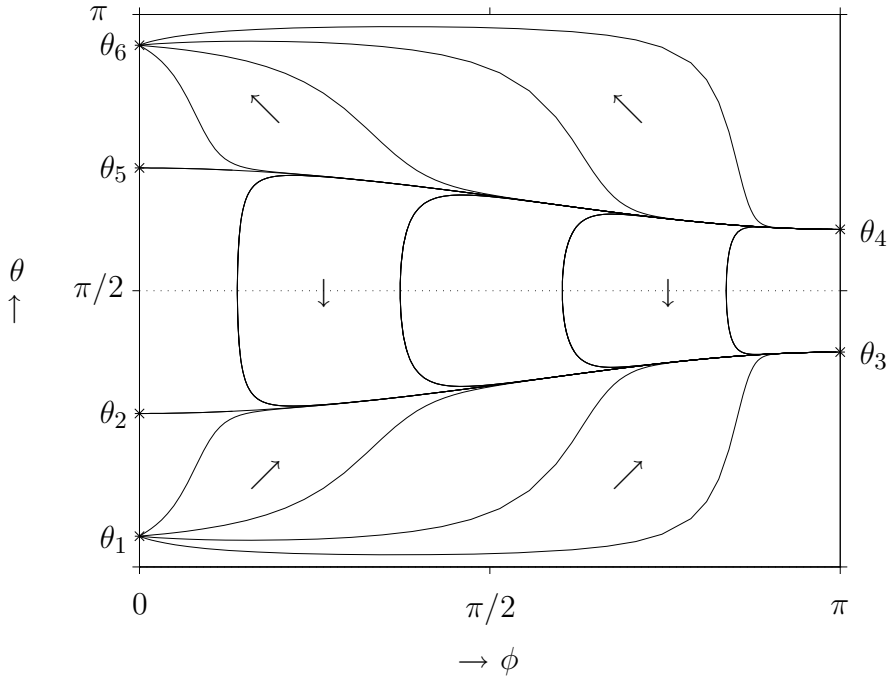


Fig. 1: The solutions of the SU(2) $n = 3$ model

The solitons interpolating amongst the vacua in $[\pi/2, \pi)$ are similar to those for the vacua in $[0, \pi/2)$ examined above. This follows immediately from the discrete symmetries (7.1) and (7.2) of the equations (4.14). These symmetries act on (θ, ϕ)

that lie on the disc constructed from the square $[0, \pi] \times [0, \pi]$ by identifying the points $(0, 0)$ and $(0, \pi)$, and the points $(\pi, 0)$ and (π, π) . For n even, the discrete symmetry is just the antipodal map of the disc leaving the centre $(\pi/2, \pi/2)$ fixed, whereas for n odd, it is a reflection at the hyperplane $\theta = \pi/2$ together with a reversal of the direction of the flow and an interchange sinks and sources.

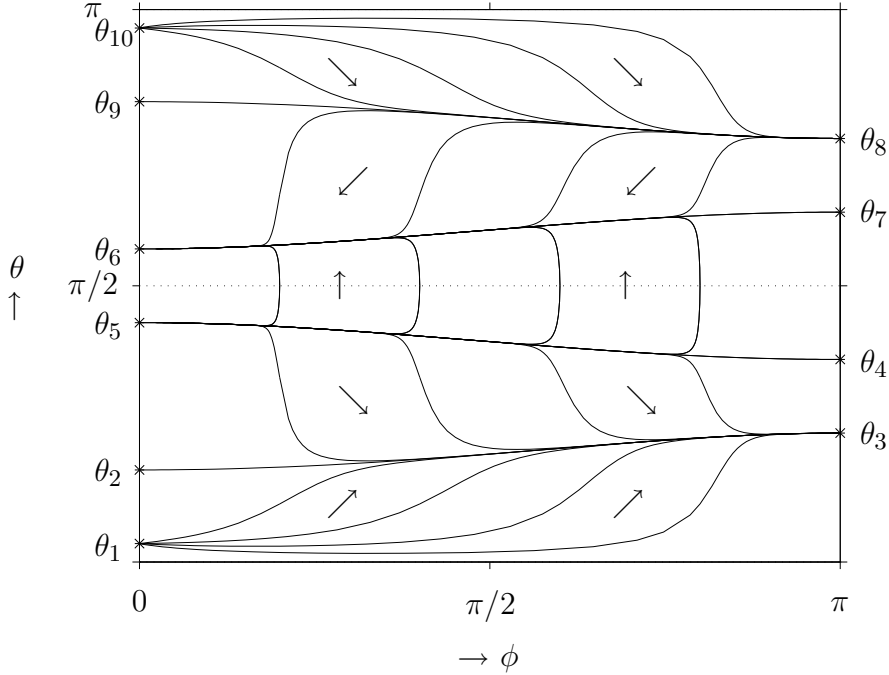


Fig. 2: The solutions of the $SU(2)$ $n = 5$ model

Let us consider the solutions amongst the vacua that lie near the boundary of $[0, \pi]$. The lowest vacuum θ_1 (which is a source) connects only to the sink θ_3 through solitons that lie in the complement of the maximal torus of $SU(2)$. Similarly the highest vacuum θ_{2n} (which is a sink) connects only to the source θ_{2n-2} . It remains to examine the solutions interpolating amongst vacua that lie ‘near’ $\theta = \pi/2$. For $n = 4\ell + 1$, $\ell \in \mathbb{N}$, the highest vacuum in $[0, \pi/2)$ is the source θ_n and this connects to the sink θ_{n-2} and to the saddle θ_{n-1} as it has been described above for the vacua that lie within $[0, \pi/2)$. In addition, θ_n connects with the lowest vacuum in $[\pi/2, \pi)$ which is the sink θ_{n+1} at the $\phi = 0$ semicircle; the solutions are similar to the ones that we have found in the $SU(2)$ $n = 1$ model in section five. The sink θ_{n+1} also connects to the source θ_{n+3} and the saddle

θ_{n+2} as required by the reflection symmetry. It is straightforward to repeat this analysis for the $n = 4\ell + 3$ case. In figures 1 and 2, we confirm our results by a numerical computation for the solutions of equation (4.14) of the $SU(2)$ $n = 3, 5$ models that lie in the complement of the maximal torus of $SU(2)$. We remark that it is possible to interpolate amongst the vacua of $SU(2)$ model with $n \geq 5$ in a way different from the one proposed above. However, the numerical calculation for $n = 5$ supports the case that has been presented.

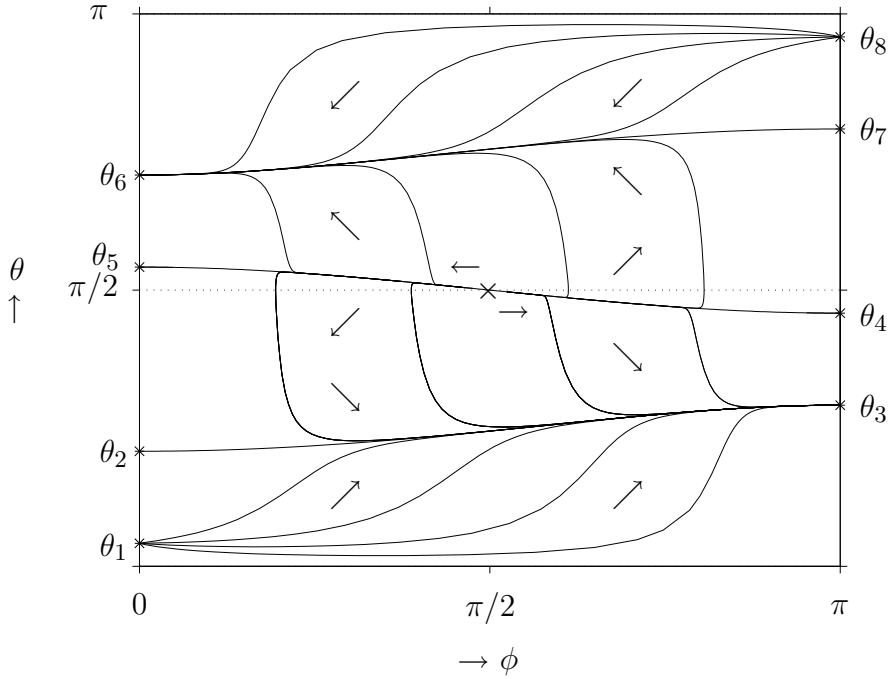


Fig. 3 : The solutions of the $SU(2)$ $n = 4$ model

Next we turn to examine the n even case and observe that $(\frac{\pi}{2}, \frac{\pi}{2})$ is a fixed point of (4.14) but *not* a vacuum of the theory; $(\frac{\pi}{2}, \frac{\pi}{2})$ is source for $n = 4\ell$ and a sink for $n = 4\ell + 2$. For $n = 4\ell + 2$, the sink θ_{n-1} connects to the saddle θ_{n-2} and to the source θ_{n-3} in the usual way. Both the sink θ_{n-1} and the saddle θ_n connect to the $(\frac{\pi}{2}, \frac{\pi}{2})$ fixed point. Using the antipodal map, we can easily determine the behaviour of the solutions associated to the vacua θ_{n+1} and θ_{n+2} from the solutions associated to the vacua θ_n and θ_{n-1} . We remark that for $n = 2$, we recover the behaviour we have found in section six. It is straightforward to repeat this analysis for the

$n = 4\ell$ case. In figure 3, we confirm our results with a numerical computation for the solutions of (4.14) of the $SU(2)$ $n = 4$ model that lie in the complement of the maximal torus of $SU(2)$.

8. The G -models

In the previous sections we have investigated the soliton solutions of sigma models with target space the groups $SO(2)$ and $SU(2)$. In this section, we shall explore some of the properties of the soliton solutions of G -models for G any semisimple compact Lie group. The vacua of any G -model can be arranged to lie on a maximal torus of the group G . To show this, we first choose, without loss of generality, κ in a Cartan subalgebra of G as in section two, i.e.

$$\kappa = \kappa^m t_m , \quad (8.1)$$

where t_m is an orthonormal basis of the Cartan subalgebra and κ^m are the components of κ ($\kappa^m \in \mathbb{R}$). Now suppose that there is a vacuum k_0 that does not lie on this maximal torus of G , then there is a $h_0 \in G$ such that $k'_0 = h_0 k_0 h_0^{-1}$ and k'_0 is in the maximal torus. Using the invariance of the equation for the vacua under conjugation, (2.13) can be written as

$$2h_0^{-1}\kappa h_0 + \sum_A t_A \text{tr}(t_A (k'_0)^n) = 0 , \quad (8.2)$$

where κ is as in (8.1). Now, this equation has solutions if and only if $h_0 \kappa h_0^{-1}$ is again in the Cartan subalgebra of G . It follows then that h_0 is an element of the Weyl group of G in which case k_0 is in the maximal torus. Therefore all vacua of the G -models are on the maximal torus of G . This allows us to set

$$k = \exp(\theta^m t_m) \quad (8.3)$$

and rewrite the equation for the vacua as

$$2\kappa_m + \text{tr}(t_m \exp(n\theta^r t_r)) = 0 . \quad (8.4)$$

Observe that the second term in the equation for the vacua is the derivative of the character $\chi(\theta) = \text{tr } k$ of G with respect to the co-ordinates of the maximal torus.

As in the $SU(2)$ model, the G -model also has two classes of solitons. (i) The static solitons that lie on the maximal torus of G . (ii) The time-dependent solitons, apart from their asymptotic values, lie in the complement of the maximal torus of G . In the former case, the equations (2.22) can be written as

$$\frac{d}{dx}\theta^m(x) = \mp \frac{m}{2} \left(2\kappa^m + \text{tr}(t^m \exp(n\theta^r t_r)) \right) . \quad (8.5)$$

A class of solutions of these equations can be found by setting all but one angle, say θ^1 , to their vacuum values. The solitons obtained in this way are precisely those that are embeddings of the $SO(2)$ solitons along the direction of θ^1 . There are as many independent directions for embedding $SO(2)$ solitons in the G -model as the rank of the group G . Apart from these solitons, there are other static solitons particular to the G -model ($\text{rank } \mathcal{L}(G) > 1$) for which more than one of the angles θ^m are not set to their vacuum values, and the equations (8.5) do not separate to a sum of $SO(2)$ ones. To see this, let us consider the $SU(3)$ model. The maximal torus in this case is T^2 and the theory has four vacua for some choice of κ . The equations (8.5) are similar to the standard Morse flow of the height function f of T^2 . Apart from the Morse flows of f which one gets by embedding the Morse flows of S^1 along the two homology cycles of T^2 and which are similar to the embedding of $SO(2)$ solitons in the $SU(3)$ model, one obtains additional flows, for example an one-parameter family of flows from the absolute maximum to the absolute minimum of f .

The investigation of the time-dependent solitons that lie in the complement of the maximal torus of G is more involved because, apart from the Bogomol'nyi

equations along the maximal torus, we should also solve the Bogomol'nyi equations along the root directions of G . Despite this, some of them can be obtained from embedding the $SU(2)$ time-dependent solitons into the G -model up to a sigma model transformation (2.24) of the G model. This is done by embedding the $SU(2)$ group into the G group up to a conjugation. This group theoretical problem has been investigated at the Lie algebra level in [11] and it was found that there is at least one embedding of $SU(2)$ in G for every positive root α of G . For such an embedding the coupling constant κ of the G -model is

$$\kappa^m = \tilde{\kappa} \alpha^m, \quad (8.6)$$

where $\tilde{\kappa}$ is a real number and $|\tilde{\kappa}| < 1/|\alpha|^2$, i.e. κ is along a direction in the Cartan subalgebra of G labeled by the root α . The generators of the $SU(2)$ subgroup of G are

$$i\tilde{h}_\alpha, \quad i(\tilde{E}_\alpha + \tilde{E}_{-\alpha}), \quad \tilde{E}_\alpha - \tilde{E}_{-\alpha}, \quad (8.7)$$

where $\{\tilde{h}_\alpha, \tilde{E}_\alpha, \tilde{E}_{-\alpha}\}$ are the generators of $\mathcal{L}(G)$ in the Chevalley basis. It is clear from (8.6) that all these embeddings do not describe the most general soliton solutions of the G -model. For this, one should solve the Bogomol'nyi equations of the G -model for κ along a generic direction in the Cartan subalgebra of G .

We expect that apart from the one-soliton solutions that we have discussed so far, the G -models may have multi-soliton solutions as well. Indeed for G -models with $\kappa = 0$, we can simply embed the multi-soliton solutions of the sine-Gordon theory (see for example [12] and references within.). However, since the multi-soliton solutions of the sine-Gordon theory are not solutions of the Bogomol'nyi equations, the same applies for embedded multi-soliton solutions in the G -model.

9. Concluding remarks and summary

Quantum mechanically, some solitons may decay to other solitons that carry the same topological charge but different Noether charge by emitting radiation. This is because, for G -models for which G is a semisimple group, the fundamental states of the theory as well as its solitons are charged with respect to the *same* Noether charge Q_N . Note that the solitons, in addition to the Noether charge Q_N , carry topological charge Q_T . From charge conservation it follows that a soliton with topological charge Q_T and with Noether charge Q_N may decay to an other soliton with the same topological charge but with Noether charge Q'_N and $|Q'_N| < |Q_N|$, and some fundamental states with Noether charge $Q_N - Q'_N$. Energy conservation though seems to rule out such a process because the fundamental states do not saturate the bound ($\kappa \neq 1$). However if for some unknown mechanism to us such a process is allowed the stable soliton configurations are those that have the least mass for given topological charge. An example of such a configuration is the soliton (3.8) of the $SU(2)$ $n = 1$ model. Moreover, as it is well known, the coupling constant of G -models, G semisimple, is quantised due to the presence of the torsion term and therefore it may be more appropriate to develop a non-perturbative method, similar to that developed for the Wess-Zumino-Witten model [13], to investigate the fundamental states of the theory instead of linearising the theory about a vacuum in the weak coupling limit. Another related issue is the quantisation of the charge Q of the solitons. Since Q is not purely topological, the quantisation of Q does not follow from classical considerations; a quantisation of the moduli space suggests though that Q should be quantised provided that the moduli coordinate ψ is periodic which is the case whenever the orbits of the vector field X in G are periodic. We also remark that even if Q is quantised the usual stability argument for BPS monopoles does not apply to this case because the bound for the energy is in terms of one charge rather than two, which are necessary to establish the stability for the solitons using the triangular inequality.

To summarise, we have investigated the soliton solutions of (1,1)-super-

symmetric massive sigma models with torsion and target space a group G for a class of scalar potentials characterised by a coupling constant κ and an integer n . These solitons are solutions of Bogomol'nyi equations which arise from the saturation of a bound for the energy of these models in terms of a charge Q that appears as a central charge in the (1,1)-supersymmetry algebra. The charge Q is the sum of a Noether charge Q_N and a topological charge Q_T . The $G = SO(2)$ model is the simplest of the G -models and its solitons can be easily computed; the $SO(2)$ model with $\kappa = 0$ is the supersymmetric sine-Gordon theory. For G a semisimple group there are two classes of solitons to consider, one is a set of static solitons that lie on a maximal torus of G and the other is a set of time-dependent solitons that, apart from their asymptotic values, lie in the complement of a maximal torus of G . For $G = SU(2)$, we have found all static solitons as embeddings of the corresponding $SO(2)$ solitons. In addition, we have explicitly computed the time-dependent solitons of the $n = 1$ model, and we have shown that all solitons of the $n = 2$ model are static. We have also presented the qualitative properties of the time-dependent solitons of the $SU(2)$ $n > 2$ models and confirmed our results with a numerical calculation for the $n = 3, 4, 5$ ones. For sigma models with target space a semisimple group G , some of their solitons can be obtained from embedding the solitons of the $SO(2)$ model and the solitons of the $SU(2)$ model with the corresponding value of n .

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